From quasi-entropy to skew information

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Abstract

This paper gives an overview about particular quasi-entropies, generalized quantum covariances, quantum Fisher informations, skew-informations and their relations. The point is the dependence on operator monotone functions. It is proven that a skew-information is the Hessian of a quasi-entropy. The skew-information and some inequalities are extended to a von Neumann algebra setting.

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1 Introductory preliminaries

Let $\mathcal{M}$ denote the algebra of $n \times n$ matrices with complex entries. For positive definite matrices $D_1, D_2 \in \mathcal{M}$, for $A \in \mathcal{M}$ and a function $F : \mathbb{R}^+ \to \mathbb{R}$, the **quasi-entropy** is defined as

$$\begin{aligned} S_F^A(D_1, D_2) &:= \langle A D_1^{1/2}, F(\Delta(D_2/D_1))(A D_1^{1/2}) \rangle \\ &= \operatorname{Tr} D_1^{1/2} A^* F(\Delta(D_2/D_1))(A D_1^{1/2}), \end{aligned} \tag{1}$$

where $\Delta(D_2/D_1) : \mathcal{M} \to \mathcal{M}$ is a linear mapping acting on matrices:

$$\Delta(D_2/D_1)A = D_2 A D_1^{-1}.$$

This concept was introduced in [21, 22], see also Chapter 7 in [20] and it is the quantum generalization of the F-entropy of Csiszár used in classical information theory (and statistics) [4, 16].

The concept of quasi-entropy includes some important special cases. If D_1 and D_2 are different and $A = I$, then we have a kind of relative entropy. For $F(x) = -\log x$ we have Umegaki's relative entropy $S(D_1 \| D_2) = \operatorname{Tr} D_1(\log D_1 - \log D_2)$. More generally,

$$F(x) = \frac{1}{\alpha(1-\alpha)}(1 - x^\alpha),$$

is operator monotone decreasing for $\alpha \in (-1, 1)$. (For $\alpha = 0$, the limit is taken and it is $-\log x$.) Then the Rényi entropies are produced

$$S_\alpha(D_1 \| D_2) := \frac{1}{\alpha(1-\alpha)} \operatorname{Tr}(I - D_2^\alpha D_1^{-\alpha}) D_1.$$

If $D_1 = D_2 = D$ and $A, B \in \mathcal{M}$ are arbitrary, then one can approach to the **generalized covariance** [25]. An operator monotone function $f : \mathbb{R}^+ \to \mathbb{R}^+$ will be called **standard** if $x f(x^{-1}) = f(x)$ and $f(1) = 1$. A standard function f admits a canonical representation

$$f(t) = e^\beta \frac{1+t}{\sqrt{2}} \exp \int_0^1 \frac{\lambda^2 - 1}{\lambda^2 + 1} \cdot \frac{1 + t^2}{(\lambda + t)(1 + \lambda t)} h(\lambda)\, d\lambda, \tag{2}$$

where $h : [0, 1] \to [0, 1]$ is a measurable function and β is a real constant [11].

If f is a standard function, then

$$\operatorname{qCov}_D^f(A, B) := \langle A D^{1/2}, f(\Delta(D/D))(B D^{1/2}) \rangle - (\operatorname{Tr} D A^*)(\operatorname{Tr} D B). \tag{3}$$

is a generalized covariance. The usual **symmetrized covariance** corresponds to the function $f(t) = (t+1)/2$:

$$\operatorname{Cov}_D(A, B) := \frac{1}{2} \operatorname{Tr}(D(A^* B + B A^*)) - (\operatorname{Tr} D A^*)(\operatorname{Tr} D B).$$

The **quantum Fisher information** is similarly defined to (1), but $F(x) = 1/f(x)$ for a standard function $f : \mathbb{R}^+ \to \mathbb{R}^+$:

$$\gamma_D^f(A,B) = \langle AD^{-1/2}, \frac{1}{f}(\Delta(D/D))(BD^{-1/2})\rangle \tag{4}$$

Quantum Fisher information was characterized by the monotonicity under coarse-graining [23]. This kind of non-affine parametrization was used in [23, 25], since the relation to operator means was emphasized. Sometimes the affine parametrization is more convenient and **Hansen's canonical representation** of the inverse of a standard operator monotone function can be used [12].

Proposition 1 *If $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a standard operator monotone function, then*

$$\frac{1}{f(t)} = \int_0^1 \frac{1+\lambda}{2}\left(\frac{1}{t+\lambda} + \frac{1}{1+t\lambda}\right) d\mu(\lambda), \tag{5}$$

where μ is a probability measure on $[0,1]$.

The theorem implies that the set $\{1/f : f \text{ is standard operator monotone}\}$ is convex and gives the extremal points

$$g_\lambda(x) := \frac{1+\lambda}{2}\left(\frac{1}{t+\lambda} + \frac{1}{1+t\lambda}\right) \qquad (0 \le \lambda \le 1). \tag{6}$$

One can compute directly that

$$\frac{\partial}{\partial\lambda} g_\lambda(x) = -\frac{(1-\lambda^2)(x+1)(x-1)^2}{2(x+\lambda)^2(1+x\lambda)^2}.$$

Hence g_λ is decreasing in the parameter λ. For $\lambda = 0$ we have the largest function $g_0(t) = (t+1)/(2t)$ and for $\lambda = 1$ the smallest is $g_1(t) = 2/(t+1)$. Note that this was also obtained in the setting of positive operator means [14], harmonic and arithmetic means.

Covariance and Fisher information are bilinear (or sesqui-linear) forms, in the applications they are mostly reduced to self-adjoint matrices.

The space $\mathcal{M}$ has an orthogonal decomposition

$$\{B \in \mathcal{M} : [D,B] = 0\} \oplus \{\mathrm{i}[D,A] : A \in \mathcal{M}\}.$$

We denote the two subspaces by $\mathcal{M}_D$ and $\mathcal{M}_D^c$, respectively. If $A_2 \in \mathcal{M}_D$, then

$$F(\Delta(D/D))(A_2 D^{\pm 1/2}) = A_2 D^{\pm 1/2}$$

implies

$$\mathrm{qCov}_D^f(A_1,A_2) = \mathrm{Tr}\, DA_1^* A_2 - (\mathrm{Tr}\, DA_1^*)(\mathrm{Tr}\, DA_2), \qquad \gamma_D^f(A_1,A_2) = \mathrm{Tr}\, D^{-1} A_1^* A_2$$

independently of the function f. Moreover, if $A_1 \in \mathcal{M}_D^c$, then

$$\gamma_D^f(A_1,A_2) = \mathrm{qCov}_D^f(A_1,A_2) = 0\,.$$

Therefore, the effect of the function f and the really quantum situation are provided by the components from $\mathcal{M}_D^c$.

2 Quasi-entropy

The quasi-entropies are monotone and jointly convex [20, 22]:

Let $\alpha : \mathcal{M}_0 \to \mathcal{M}$ be a mapping between two matrix algebras. The dual $\alpha^* : \mathcal{M} \to \mathcal{M}_0$ with respect to the Hilbert-Schmidt inner product is positive if and only if α is positive. Moreover, α is unital if and only if α^* is trace preserving. $\alpha : \mathcal{M}_0 \to \mathcal{M}$ is called a **Schwarz mapping** if

$$\alpha(B^*B) \geq \alpha(B^*)\alpha(B) \tag{7}$$

for every $B \in \mathcal{M}_0$.

Proposition 2 *Assume that $F : \mathbb{R}^+ \to \mathbb{R}$ is an operator monotone function with $F(0) \geq 0$ and $\alpha : \mathcal{M}_0 \to \mathcal{M}$ is a unital Schwarz mapping. Then*

$$S_F^A(\alpha^*(D_1), \alpha^*(D_2)) \geq S_F^{\alpha(A)}(D_1, D_2) \tag{8}$$

holds for $A \in \mathcal{M}_0$ and for invertible density matrices D_1 and D_2 from the matrix algebra $\mathcal{M}$.

If we apply the monotonicity (8) to the embedding $\alpha(X) = X \oplus X$ of $\mathcal{M}$ into $\mathcal{M} \oplus \mathcal{M}$ and to the densities $D_1 = \lambda E_1 \oplus (1-\lambda)F_1$, $D_2 = \lambda E_2 \oplus (1-\lambda)F_2$, then we obtain the joint concavity of the quasi entropy.

Proposition 3 *Under the conditions of Theorem 2, the joint concavity*

$$\lambda S_F^A(E_1, E_2) + (1-\lambda)S_F^A(F_1, F_2) \leq S_F^A(\lambda E_1 + (1-\lambda)F_1, \lambda E_2 + (1-\lambda)F_2) \tag{9}$$

holds.

The case $F(t) = t^\alpha$ is the famous Lieb's concavity theorem [15].

Our next aim is to compute

$$\frac{\partial^2}{\partial t \partial s} S_F(D + tA, D + sB)\Big|_{t=s=0}. \tag{10}$$

We shall use the formulas

$$\frac{d}{dt} h(D + tB)\Big|_{t=0} = Bh'(D) \quad (B \in \mathcal{M}_D), \qquad \frac{d}{dt} h(D + t\mathrm{i}[D, X])\Big|_{t=0} = \mathrm{i}[h(D), X]\,,$$

see [26].

Lemma 1 *If $A, B \in \mathcal{M}_D$, then the derivative (10) equals $-F''(1)\mathrm{Tr}\, D^{-1}AB$.*

Proof: It is enough to check the case $F(t) = t^n$. Then

$$S_F(D + tA, D + sB) = \mathrm{Tr}\,(D + tA)^{1-n}(D + sB)^n \tag{11}$$

and the derivative is $(1 - n)n\mathrm{Tr}\, D^{-1}AB$. □

Lemma 2 *If $A \in \mathcal{M}_D$ and $B \in \mathcal{M}_D^c$, then the derivative (10) equals* 0.

Proof: We compute for $F(t) = t^n$ using (11). If $B = [D, X]$, then we have the derivative

$$\mathrm{Tr}\,(1 - n)D^{-n}A[D^n, X] = 0$$

and this gives the statement. □

Lemma 3 *Let $X = X^* \in \mathcal{M}$ and $F : \mathbb{R}^+ \to \mathbb{R}$ be a continously differentiable function. Then*

$$\frac{\partial^2}{\partial t \partial s} S_F(D + t\mathrm{i}[D, X], D + s\mathrm{i}[D, X])\Big|_{t=s=0} = 2F(1)\mathrm{Tr}\, DX^2 - 2S_F^X(D, D). \tag{12}$$

Proof: Since both sides are linear and continuous in F, we may assume that $F(t) = t^n$. Derivation of formula (11) gives

$$2\mathrm{Tr}\, DX^2 - 2\mathrm{Tr}\, XD^{1-n}XD^n$$

and this is the stated result for the particular F. □

3 Skew information

The Wigner-Yanase-Dyson skew information is the quantity

$$I_p(D, A) := -\frac{1}{2}\mathrm{Tr}\,[D^p, A][D^{1-p}, A] \qquad (0 < p < 1).$$

Actually, the case $p = 1/2$ is due to Wigner and Yanase [29] and the extension was proposed by Dyson. The convexity of $I_p(D, A)$ in A is a famous result of Lieb [15]

It was observed in [24] that the Wigner-Yanase-Dyson skew information is connected to a monoton Riemannian metric (or Fisher information) which corresponds to the function

$$f_p(x) = p(1 - p)\,\frac{(x - 1)^2}{(x^p - 1)(x^{1-p} - 1)}\,.$$

It was proven in [24] that this is an operator monotone function, a generalization was obtained in [12, 28].

Let f be a standard function and $X = X^* \in \mathcal{M}$. The quantity

$$I_D^f(X) := \frac{f(0)}{2} \gamma_D^f(\mathrm{i}[D,X], \mathrm{i}[D,X])$$

was called **skew information** in [12] in this general setting. Note that the parametrization in [12] is by $c = 1/f$ which is called there Morozova-Chentsov function. The skew information is nothing else but the Fisher information restricted to $\mathcal{M}_D^c$, but it is parametrized by the commutator. Skew information appears, for example, in uncertainty relations [1, 5, 6, 7, 13, 17, 18], see also Theorem 4. In that application, the skew information is regarded as a bilinear form.

If $D = \mathrm{Diag}\,(\lambda_1, \dots, \lambda_n)$ is diagonal, then

$$\gamma_D^f(\mathrm{i}[D,X], \mathrm{i}[D,X]) = \sum_{ij} \frac{(\lambda_i - \lambda_j)^2}{\lambda_j f(\lambda_i/\lambda_j)} |X_{ij}|^2.$$

This implies that the identity

$$f(0)\gamma_D^f(\mathrm{i}[D,X], \mathrm{i}[D,X]) = 2\mathrm{Cov}_D(X,X) - 2\mathrm{qCov}_D^{\tilde{f}}(X,X) \tag{13}$$

holds if $\mathrm{Tr}\, DX = 0$ and

$$\tilde{f}(x) := \frac{1}{2}\left((x+1) - (x-1)^2 \frac{f(0)}{f(x)} \right). \tag{14}$$

Since the right-hand-sides of (12) and (13) are the same if $F = \tilde{f}$ we have

Theorem 1 *Assume that $X = X^* \in \mathcal{M}$ and $\mathrm{Tr}\, DX = 0$. If f is a standard function such that $f(0) \neq 0$, then*

$$\frac{\partial^2}{\partial t \partial s} S_F(D + t\mathrm{i}[D,X], D + s\mathrm{i}[D,X])\Big|_{t=s=0} = f(0)\gamma_D^f(\mathrm{i}[D,X], \mathrm{i}[D,X])$$

for the standard function $F = \tilde{f}$.

The only remaing thing to show is that if $f : \mathbb{R}^+ \to \mathbb{R}$ is a standard function, then $\tilde{f}$ is standard as well. This result appeared in [8] and the proof there is not easy, even matrix convexity of functions of two variables is used. Here we give a rather elementary proof based on the fact that $1/f \mapsto \tilde{f}$ is linear and on the canonical decomposition in Theorem 1.

Lemma 4 *Let $\lambda \geq 0$ and $f : \mathbb{R}^+ \to \mathbb{R}$ be a function such that*

$$\frac{1}{f(x)} := \frac{1+\lambda}{2}\left(\frac{1}{x+\lambda} + \frac{1}{1+x\lambda} \right) = g_\lambda(x).$$

Then the function $\widetilde{f} : \mathbb{R}^+ \to \mathbb{R}$ defined in (14) is an operator monotone standard function.

Proof: From the definitions we obtain

$$\widetilde{f}(x)=\frac{x(x\lambda^2+\lambda^2+2\lambda+2x\lambda+x+1)}{2(x+\lambda)(1+x\lambda)}$$

and

$$\widetilde{f}'(x)=\frac{\lambda+2x\lambda+2\lambda^2+x^2\lambda+4x\lambda^2+2\lambda^3x+x^2+\lambda^3x^2+\lambda^3+\lambda^4x^2}{2(x+\lambda)^2(1+x\lambda)^2}.$$

Hence $\widetilde{f}(0)=0$ and $\widetilde{f}:\mathbb{R}^+\to\mathbb{R}^+$. So it is enough to prove that the holomorphic extension of $\widetilde{f}$ to the complex upper half-plane maps the upper half-plane into itself, see [3].

Let $a,b\in\mathbb{R}$, $b>0$. Then we have

$$\begin{aligned}\operatorname{Im}\widetilde{f}(a+ib) &= \frac{b}{2((a+\lambda)^2+b^2)((1+\lambda a)^2+\lambda^2b^2)}\\ &\quad\times(\lambda+2\lambda^2+\lambda^3+b^2+a^2+2\lambda a+\lambda a^2\\ &\qquad+4\lambda^2a+\lambda b^2+\lambda^3a^2+\lambda^3b^2+2\lambda^3a+\lambda^4a^2+\lambda^4b^2).\end{aligned}$$

Here

$$\begin{aligned}&\lambda+2\lambda^2+\lambda^3+b^2+a^2+2\lambda a+\lambda a^2+4\lambda^2a+\lambda b^2+\lambda^3a^2+\lambda^3b^2+2\lambda^3a+\lambda^4a^2+\lambda^4b^2\\ &\quad=(1+\lambda+\lambda^4+\lambda^3)a^2+(4\lambda^2+2\lambda+2\lambda^3)a+\lambda+2\lambda^2+\lambda^3+b^2(1+\lambda+\lambda^4+\lambda^3).\end{aligned}$$

The function $g:\mathbb{R}\to\mathbb{R}$

$$g(a):=(1+\lambda+\lambda^4+\lambda^3)a^2+(4\lambda^2+2\lambda+2\lambda^3)a+\lambda+2\lambda^2+\lambda^3$$

has a minimum value at

$$a(\lambda)=-\frac{4(\lambda^2+2\lambda+2\lambda^3)}{2(1+\lambda+\lambda^4+\lambda^3)}$$

and

$$g(a(\lambda))=\frac{(\lambda^2-1)^2\lambda}{(\lambda-1/2)^2+3/4}\geq 0.$$

Therefore the upper half-plane is mapped into itself. The properties $x\widetilde{f}(x^{-1})=\widetilde{f}(x)$ and $\widetilde{f}(1)=1$ are obvious. □

The uncertainty relation recently obtained is the following [9].

Proposition 4 *Assume that $f,g:\mathbb{R}^+\to\mathbb{R}$ are standard functions and D is a positive definite matrix. Then for self-adjoint matrices $A_1,A_2,\dots,A_m$ the determinant inequality*

$$\begin{aligned}&\operatorname{Det}\Big([\mathrm{qCov}_D^g(A_i,A_j)]_{i,j=1}^m\Big)\\ &\qquad\geq\operatorname{Det}\Big(\Big[f(0)g(0)(\mathrm{Cov}_D(A_i,A_j)-\mathrm{qCov}_D^{\tilde f}(A_i,A_j))\Big]_{i,j=1}^m\Big)\end{aligned}$$

holds.

Note that the right-hand-side contains skew informations, cf. (13).

4 The setting of von Neumann algebras

Let $\mathcal{M}$ be a von Neumann algebra. Assume that it is in standard form, it acts on a Hilbert space $\mathcal{H}$, $\mathcal{P} \subset \mathcal{H}$ is the positive cone and $J : \mathcal{H} \to \mathcal{H}$ is the modular conjugation [10, 20, 27]. Let φ and ω be normal states with representing vectors Φ and Ω in the positive cone. For the sake of simplicity, assume that φ and ω are faithful. This means that Φ and Ω are cyclic and separating vectors. The closure of the unbounded operator $A\Omega \mapsto A^*\Phi$ has a polar decomposition $J\Delta(\varphi,\omega)^{1/2}$ and $\Delta(\varphi,\omega)$ is called relative modular operator. $A\Omega$ is in the domain of $\Delta(\varphi,\omega)^{1/2}$ for every $A \in \mathcal{M}$.

For $A \in \mathcal{M}$ and $F : \mathbb{R}^+ \to \mathbb{R}$, the quasi-entropy

$$S_F^A(\omega,\varphi) := \langle A\Omega, F(\Delta(\varphi,\omega))A\Omega\rangle \tag{15}$$

was introduced in [21], see also Chapter 7 in [20]. (The right-hand-side can be understood via the spectral decomposition of the positive operator $\Delta(\varphi,\omega)$.) For $F(t) = -\log t$ and $A = I$ the relative entropy of Araki is obtained [2] and this was the motivation of the generalization.

Theorem 2 *Assume that $F : \mathbb{R}^+ \to \mathbb{R}$ is an operator monotone function with $F(0) \geq 0$ and $\alpha : \mathcal{M}_0 \to \mathcal{M}$ is a unital normal Schwarz mapping. Then*

$$S_F^A(\omega \circ \alpha, \varphi \circ \alpha) \geq S_F^{\alpha(A)}(\omega,\varphi) \tag{16}$$

holds for $A \in \mathcal{M}_0$ and for normal states ω and φ of the von Neumann algebra $\mathcal{M}$.

We sketch the proof based on inequalities for operator monotone and operator concave functions. (The details are clarified in [21].) First note that

$$S_{F+c}^A(\omega \circ \alpha, \varphi \circ \alpha) = S_F^A(\omega \circ \alpha, \varphi \circ \alpha) + c\,\omega(\alpha(A^*A))$$

and

$$S_{F+c}^{\alpha(A)}(\omega,\varphi) = S_F^{\alpha(A)}(\omega,\varphi) + c\,\omega(\alpha(A)^*\alpha(A))$$

for a positive constant c. Due to the Schwarz inequalty (7), we may assume that $F(0) = 0$.

Let Ω_0 be the representing vector for $\omega \circ \alpha$ and $\Delta := \Delta(\varphi,\omega)$, $\Delta_0 := \Delta(\varphi \circ \alpha, \omega \circ \alpha)$. The operator

$$Vx\Omega_0 = \alpha(x)\Omega \qquad (x \in \mathcal{M}_0) \tag{17}$$

is a contraction:

$$\|\alpha(x)\Omega\|^2 = \omega(\alpha(x)^*\alpha(x)) \leq \omega(\alpha(x^*x) = \|x\Omega_0\|^2$$

since the Schwarz inequality is applicable to α. A similar simple computation gives that

$$V^*\Delta V \leq \Delta_0\,. \tag{18}$$

Since F is operator monotone, we have $F(\Delta_0) \geq F(V^*\Delta V)$. Recall that F is operator concave, therefore $F(V^*\Delta V) \geq V^*F(\Delta)V$ and we conclude

$$F(\Delta_0) \geq V^*F(\Delta)V\,. \tag{19}$$

Application to the vector $A\Omega_0$ gives the inequality.

The natural extension of the covariance (from probability theory) is

$$\mathrm{qCov}^f_\omega(A,B) = \langle \sqrt{f(\Delta(\omega,\omega))}A\Omega, \sqrt{f(\Delta(\omega,\omega))}B\Omega\rangle - \overline{\omega(A)}\omega(B), \tag{20}$$

where $\Delta(\omega,\omega)$ is actually the modular operator. Motivated by the application, we always assume that the function f is standard. For such a function f, the inequalities

$$\frac{2x}{x+1} \leq f(x) \leq \frac{1+x}{2}$$

holds. Therefore $A\Omega$ is in the domain of $\sqrt{f(\Delta(\omega,\omega))}$ and the covariance $\mathrm{qCov}^f_\omega(A,B)$ is a well-defined sesquilinear form.

For a standard function $f : \mathbb{R}^+ \to \mathbb{R}^+$ and for a normal unital Schwarz mapping $\beta : \mathcal{N} \to \mathcal{M}$ the inequality

$$\mathrm{qCov}^f_\omega(\beta(X),\beta(X)) \leq \mathrm{qCov}^f_{\omega\circ\beta}(X,X) \qquad (X \in \mathcal{N}) \tag{21}$$

is a particular case of Theorem 2 and it is the monotonicity of the generalized covariance under coarse-graining [25].

Following [12], the skew information (as a bilinear form) can be defined as

$$I^f_\omega(X,Y) := \mathrm{Cov}_\omega(X,Y) - \mathrm{qCov}^{\tilde{f}}_\omega(X,Y) \tag{22}$$

if $\omega(X) = \omega(Y) = 0$. (Then $I^f_\omega(X) = I^f_\omega(X,X)$.)

Theorem 3 *Assume that $f,g : \mathbb{R}^+ \to \mathbb{R}$ are standard functions and ω is a faithful normal state on a von Neumann algebra $\mathcal{M}$. Let $A_1, A_2, \dots, A_m \in \mathcal{M}$ be self-adjoint operators such that $\omega(A_1) = \omega(A_2) = \ldots = \omega(A_m) = 0$. Then the determinant inequality*

$$\mathrm{Det}\Big([\mathrm{qCov}^g_D(A_i,A_j)]_{i,j=1}^m\Big) \geq \mathrm{Det}\Big(\big[2g(0)I^f_\omega(A_i,A_j)\big]_{i,j=1}^m\Big) \tag{23}$$

holds.

Proof: Let $E(\,\cdot\,)$ be the spectral measure of $\Delta(\omega,\omega)$. Then for $m = 1$ the inequality is

$$\int g(\lambda)\,d\mu(\lambda) \leq g(0)\left(\int \frac{1+\lambda}{2}\,d\mu(\lambda) - \int \tilde{f}(\lambda)\,d\mu(\lambda)\right),$$

where $d\mu(\lambda) = d\langle A\Omega, E(\lambda)A\Omega\rangle$. Since the inequality

$$f(x)g(x) \geq f(0)g(0)(x-1)^2 \tag{24}$$

holds for standard functions [9], we have

$$g(\lambda) \geq g(0)\left(\frac{1+\lambda}{2} - f(0)\tilde{f}(\lambda)\right)$$

and this implies the integral inequality.

Consider the finite dimensional subspace $\mathcal{N}$ generated by the operators $A_1, A_2, \dots, A_m$. On $\mathcal{N}$ we have the inner products

$$\langle\langle A, B\rangle\rangle := \mathrm{Cov}^g_\omega(A, B)$$

and

$$\langle A, B\rangle := 2g(0)I^f_\omega(A, B).$$

Since $\langle A, A\rangle \leq \langle\langle A, A\rangle\rangle$, the determinant inequality holds (see Lemma 2 in [9]). $\square$

This theorem is interpreted as quantum uncertainty principle [1, 6, 8, 13]. In the earlier works the function g from the left-hand-side was $(x+1)/2$ and the proofs were more complicated. The general g appeared in [9].

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